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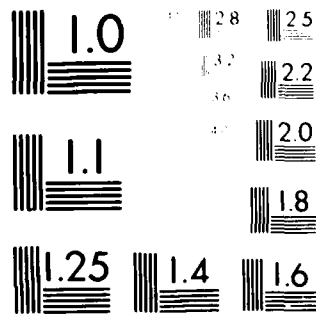   MAR 80   M F NEUTS   AFOSR-77-3236
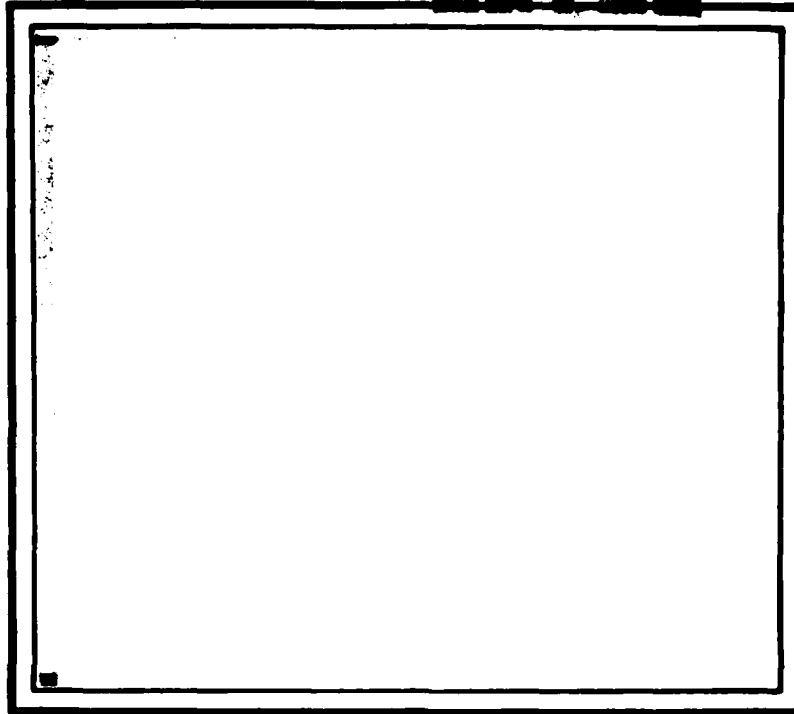UNCLASSIFIED   TR-53B   AFOSR-TR-80-0635   NL

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EXPLICIT STEADY-STATE SOLUTIONS TO SOME

ELEMENTARY QUEUEING MODELS

by

Marcel F. Neuts

Department of Mathematical Sciences

Newark, DE 19711

Applied Mathematics Institute

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Abstract

Several simple queueing models, which are commonly used as text book examples, have explicit steady-state solutions, which to date have escaped notice. These solutions are easily computable. In addition to their didactic interest, the results, presented here, are also useful in the analysis of more complex queueing models.

Key Words

Theory of Queues, phase type distributions, computational probability, finite queues, explicit solutions.

1. Introduction

The queueing models $M/E_m/1$ and $E_m/M/c$ and, to a lesser extent, their bounded versions $M/E_m/1/K+1$ and $E_m/M/c/K+c$ are familiar topics in texts dealing with the theory of queues. The analysis is mostly based on transform methods. Even when clearly and succinctly presented, as e.g. in Kleinrock [1], this analysis involves a number of formal steps, which are difficult to the student lacking in mathematical maturity. The same comments apply to the discussions of the $M/H_m/1$ queue and similar models, in which the Erlang distribution is replaced by the hyperexponential distribution. The number of journal articles, devoted to these particular queues, is very large. A substantial number of references may be found in [5]; they will be omitted here.

The purpose of this paper is to show that, even with E_m replaced by <u>a probability distribution of phase type</u> (PH) [2, 3, 5], these queues have highly tractable steady-state solutions. Except in the case of the unbounded PH/M/c queue, these solutions are fully explicit and require only the evaluation of one or more readily computed matrix inverses.

We recall that a PH-distribution $F(\cdot)$ with representation $(\underline{\alpha}, T)$ is the distribution of the time till absorption in the in the $(m + 1)$ - state Markov process with generator

$$\begin{vmatrix} T & \underline{T}^o \\ \underline{0} & 0 \end{vmatrix}.$$

and initial probability vector $(\underline{\alpha}, \alpha_{m+1})$. The square matrix T is nonsingular, has negative diagonal elements, nonnegative off-diagonal elements and satisfies $T\underline{e} + \underline{T}^O = \underline{0}$. In order to avoid trivial considerations, we shall assume that $\alpha_{m+1} = 0$, so that the row vector $\underline{\alpha}$ is a probability vector. We further assume (without loss of generality) that the matrix $T + \underline{T}^O \cdot \underline{\alpha}$ is irreducible. The representation $(\underline{\alpha}, T)$ is then said to be irreducible.

The mean μ_1' of $F(\cdot)$ is given by $\mu_1' = - \underline{\alpha} T^{-1} \underline{e}$. The irreducible generator $T + \underline{T}^O \cdot \underline{\alpha}$ has a unique, positive stationary vector $\underline{\pi}$, which is given by

$$(1) \qquad \underline{\pi} = - \mu_1' \underline{\alpha} T^{-1}.$$

The probability distribution $F(\cdot)$ itself is given by

$$(2) \qquad F(x) = 1 - \underline{\alpha} \exp(Tx)\underline{e}, \qquad \text{for } x \geq 0,$$

and it, as well as its density, are readily computed by the numerical solution of the system of differential equations

$$\underline{v}'(x) = \underline{v}(x)T, \qquad \text{for } x \geq 0,$$

$$\underline{v}(0) = \underline{\alpha},$$

$$(3)$$

$$F(x) = 1 - \underline{v}(x)\underline{e},$$

$$F'(x) = \underline{v}(x)\underline{T}^O, \qquad \text{for } x \geq 0.$$

The familiar (generalized) Erlang and hyperexponential distributions are particular PH-distributions, respectively with representations

$$
T = \begin{vmatrix}
-\lambda_1 & \lambda_1 & & & \\
& -\lambda_2 & \lambda_2 & & \\
& & \cdots & & \\
& & & & -\lambda_m
\end{vmatrix}, \quad \underline{\alpha} = (1, 0, \ldots, 0),
$$

with $\lambda_j > 0$, for $1 \le j \le m$, and

$$
T = \text{diag} \ (-\lambda_1, \ldots, -\lambda_m), \qquad \underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m),
$$

with $\lambda_j > 0$, $\alpha_j > 0$, for $1 \le j \le m$.

The following lemmas are basic to the sequel.

Lemma 1

Let $(\underline{\alpha}, T)$ be an irreducible representation, then for $\lambda > 0$, then matrix $\lambda I - \lambda \underline{e} \cdot \underline{\alpha} - T$ is nonsingular.

Its inverse, which is given by

$$
(4) \quad (\lambda I - \lambda \underline{e} \cdot \underline{\alpha} - T)^{-1} = (\lambda I - T)^{-1} + \frac{\lambda}{f(\lambda)} (\lambda I - T)^{-1} \underline{e} \cdot \underline{\alpha} (\lambda I - T)^{-1},
$$

is strictly positive. The quantity $f(\lambda)$ is equal to the Laplace-Stieltjes transform $f(s) = \underline{\alpha}(sI - T)^{-1} \underline{T}^0$, of $F(\cdot)$, evaluated at $s = \lambda$.

Proof

If the matrix $\lambda I - \lambda \underline{e} \cdot \underline{\alpha} - T$ were singular, there would exist a vector $\underline{u} \neq \underline{0}$, such that

$$(5) \qquad \underline{u}\, T + \lambda(\underline{u}\,\underline{e})\,\underline{\alpha} - \lambda\,\underline{u} = \underline{0}.$$

If $\underline{u}\,\underline{e} = 0$, then the nonsingularity of $\lambda I - T$ implies that $\underline{u} = \underline{0}$. As this is a contradiction, we may normalize $\underline{u}$ by setting $\underline{u}\,\underline{e} = 1$. Postmultiplying by $\underline{e}$ in (5) now yields that $\underline{u}\,\underline{T}^O = 0$, and that $\underline{u} = \lambda\underline{\alpha}(\lambda I - T)^{-1}$. It follows that

$$\underline{u}\,\underline{T}^O = \lambda\,\underline{\alpha}(\lambda I - T)^{-1}\underline{T}^O = \lambda\,f(\lambda) = 0,$$

which is clearly impossible. The vector $\underline{u}$ is therefore the zero vector and the matrix is nonsingular.

We may now write

$$(\lambda I - \lambda\underline{e}\cdot\underline{\alpha} - T)^{-1} = (\lambda I - T)^{-1}\left[I - \lambda\underline{e}\cdot\underline{\alpha}(\lambda I - T)^{-1}\right]^{-1}$$

$$= (\lambda I - T)^{-1}\left\{I + \sum_{\nu=1}^{\infty}\left[\lambda\,\underline{e}\cdot\underline{\alpha}(\lambda I - T)^{-1}\right]^{\nu}\right\}$$

$$= (\lambda I - T)^{-1}\left\{I + \lambda\,\underline{e}\cdot\underline{\alpha}\,(\lambda I - T)^{-1}\sum_{\nu=0}^{\infty}\left[\underline{\alpha}\,(\lambda I - T)^{-1}\,\underline{e}\right]^{\nu}\right\}.$$

It is immediate from (2) that $\underline{\alpha}(\lambda I - T)^{-1}\underline{e} = 1 - f(\lambda)$, so that Formula (4) follows by elementary manipulations. By using classical properties of differential equations, it was shown in [3] that the vectors $\underline{\alpha}\,\exp(Tx)$ and $\exp(Tx)\,\underline{T}^O$, are positive

for $x > 0$. This implies that the vectors $\underline{\alpha}(\lambda I - T)^{-1}$ and

$(\lambda I - T)^{-1} \underline{T}^0$ are positive. Since $(\lambda I - T)^{-1}$ is nonnegative,

we conclude that the inverse in (4) is a positive matrix.

Lemma 2

The maximal eigenvalue η of the positive matrix

(6) $R = \lambda(\lambda I - \lambda\underline{e}\cdot\underline{\alpha} - T)^{-1}$,

is given by $\eta = \lambda(\lambda - c^*)^{-1}$, where c^* is the unique real
solution of the equation

(7) $\lambda \underline{\alpha}(cI - T)^{-1} \underline{e} = 1$.

Moreover, if $-\tau$ is the abscissa of convergence of the transform
$f(s)$, then $-\tau < c^* < \lambda$. The eigenvalue η is less than one, if
and only if $\lambda \mu_1' < 1$.

Proof.

Let $\underline{u}$ be the positive left eigenvector of R , corresponding
to its Perron-Frobenius eigenvalue η and let $\underline{u}$ be normalized
by $\underline{u} \underline{e} = 1$. The equation $\underline{u} R = \eta \underline{u}$, leads to
$\lambda \underline{u} = \lambda \eta \underline{u} - \lambda \eta \underline{\alpha} - \eta \underline{u} T$, and the latter equation leads, upon
postmultiplication by $\underline{e}$, to $\eta = \frac{\lambda}{d}$, where $d = \underline{u} \underline{T}^0 > 0$.

Upon substitution into the preceding equation for $\underline{u}$, we
obtain $\underline{u} = \lambda \underline{\alpha} \left[(\lambda - d) I - T\right]^{-1}$. Since $\underline{u} \underline{e} = 1$, we now readily
obtain the equation (7).

The function $\phi(c)$ is strictly decreasing, positive and convex on $(-\tau, \infty)$ and has a pole at $c = -\tau$. It tends to zero as c tends to infinity. The equation (7) therefore has a unique real solution. If $\lambda\mu_1' < 1$, then $\phi(0) = \lambda\mu_1' < 1$, so that $c^* < 0$, and therefore $0 < \eta < 1$. If $\lambda\mu_1' = 1$, we obtain $\phi(0) = 1$, and hence $\eta = 1$. If $\lambda\mu_1' = \phi(0) > 1$, then c^* is positive and $\eta > 1$. Finally $\lambda\underline{\alpha}(\lambda I - T)^{-1}\underline{e} = 1 - f(\lambda) < 1$, so that $c^* < \lambda$.

2. The M/PH/1 and M/PH/1/K+1 Queues

The M/PH/1 queue with Poisson arrival rate λ may be studied as a Markov process with the state space $E = \{0, (i, j), i \geq 1, 1 \leq j \leq m\}$, where the state 0 corresponds to the empty queue and (i, j) indicates that there are $i \geq 1$ customers present with the service in course in its phase j.

The generator Q of that Markov process is given by

$$(8) \quad Q = \begin{array}{c} 0 \\ \underline{1} \\ \underline{2} \\ \underline{3} \\ \underline{4} \\ \vdots \end{array} \left[\begin{array}{ccccc} -\lambda & \lambda\underline{\alpha} & \underline{0} & \underline{0} & \cdots \\ \underline{T}^O & T-\lambda I & \lambda I & 0 & \cdots \\ \underline{0} & \underline{T}^O \cdot \underline{\alpha} & T-\lambda I & \lambda I & \cdots \\ 0 & 0 & \underline{T}^O \cdot \underline{\alpha} & T-\lambda I & \cdots \\ \underline{0} & 0 & 0 & \underline{T}^O \cdot \underline{\alpha} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \right] ,$$

in block-partitioned form. The pair $(\underline{\alpha}, T)$ is here the irreducible representation of the service time distribution $F(\cdot)$. By $\underline{i}$, we

denote the set of states $\{(i, j), 1 \le j \le m\}$.

The stationary probability vector $\underline{x}$ of Q is partitioned into x_0, $\underline{x}_1$, $\underline{x}_2$, ..., where the vectors $\underline{x}_i$, $i \ge 1$, are of dimension m. It is explicitly given by the following theorem.

Theorem 1

Provided that $\rho = \lambda \mu_1' < 1$, we have

$$x_0 = 1 - \rho,$$

(9)

$$\underline{x}_i = (1 - \rho) \underline{\alpha} R^i, \qquad \text{for } i \ge 1,$$

where R is the matrix, defined in (6).

Proof

The equation $\underline{x} \, Q = \underline{0}$, is equivalent to

$$-\lambda x_0 + \underline{x}_1 \, \underline{T}^o = 0,$$

(10)
$$\lambda x_0 \underline{\alpha} + \underline{x}_1 (T - \lambda I) + (\underline{x}_2 \, \underline{T}^o) \, \underline{\alpha} = \underline{0},$$

$$\lambda \underline{x}_{i-1} + \underline{x}_i (T - \lambda I) + (\underline{x}_{i+1} \, \underline{T}^o) \, \underline{\alpha} = 0, \quad \text{for } i \ge 2.$$

Postmultiplying each equation, but the first, by $\underline{e}$ and recalling that $T\underline{e} = - \underline{T}^o$, we obtain

$$\underline{x}_1 \, T^o = \lambda x_0,$$

$$\underline{x}_{i+1} \, T^o = \lambda \underline{x}_i \underline{e}, \qquad \text{for } i \ge 1.$$

Substitution into the equation (10), we obtain that

$$\underline{x}_1 (\lambda I - \lambda \underline{e} \cdot \underline{\alpha} - T) = \lambda x_0 \underline{\alpha}, \text{ and } \underline{x}_{i+1}(\lambda I - \lambda \underline{e} \cdot \underline{\alpha} - T) = \lambda \underline{x}_i,$$

for $i \geq 1$.

It now only remains to verify that $x_0 = 1 - \rho$. Since the normalizing equation reduces to

$$x_0 + \sum_{i=1}^{\infty} x_0 \underline{\alpha} R^i \underline{e} = x_0 + x_0 \underline{\alpha} R(I - R)^{-1} \underline{e} = x_0 \underline{\alpha}(I - R)^{-1} \underline{e} = 1,$$

we see that the queue is <u>stable</u> if and only if $\rho < 1$.
The preceding equation yields upon substitution for R that

$$x_0 \underline{\alpha}(I - R)^{-1} \underline{e} = x_0 \underline{\alpha} T(\lambda \underline{e} \cdot \underline{\alpha} + T)^{-1} = x_0 \underline{\alpha}(I + \lambda \underline{e} \cdot \underline{\alpha} T^{-1})^{-1} \underline{e}$$

$$= x_0 + x_0 \sum_{\nu=1}^{\infty} (-1)^{\nu} \lambda^{\nu} \underline{\alpha} \left[\underline{e} \cdot \underline{\alpha} T^{-1}\right]^{\nu} \underline{e} = x_0 + x_0 \rho (1-\rho)^{-1} = 1,$$

since $\mu'_1 = - \underline{\alpha} T^{-1} \underline{e}$. This completes the proof of the theorem.

Corollary 1.

The stationary queue length density at an arbitrary time is given by

$$y_0 = 1 - \rho,$$

$$y_i = (1-\rho) \underline{\alpha} R^i \underline{e}, \qquad \qquad \text{for } i \geq 1.$$

This is also the stationary density of the queue length following departures.

Proof

Only the second statement requires proof. The probability that i customers remain following a departure is given by

$$\frac{\underline{x}_{i+1}\,\underline{T}^O}{\sum\limits_{\nu=1}^{\infty}\underline{x}_\nu\underline{T}^O} = \frac{\underline{x}_{i+1}\underline{T}^O}{(1-\rho)\,\underline{\alpha}\,(I-R)^{-1}R\,\underline{T}^O} = y_i, \qquad \text{for } i \geq 0$$

since $R\,\underline{T}^O = \lambda\,\underline{e}$, and $(1-\rho)\,\underline{\alpha}\,(I-R)^{-1}\,\underline{e} = 1$.

Corollary 2

Given that there are $i \geq 1$, customers in the stationary queue at an arbitrary time t, the conditional distribution of the residual service time is of phase type with representation $(\underline{\beta}_i, T)$, with $\underline{\beta}_i$ given by

$$\underline{\beta}_i = (\underline{\alpha}\,R^i\,\underline{e})^{-1}\,\underline{\alpha}\,R^i, \qquad \text{for } i \geq 1.$$

The mean of this distribution is given by

$$\gamma_i = -(\underline{\alpha}\,R^i\,\underline{e})^{-1}(\underline{\alpha}\,R^i\,T^{-1}\,\underline{e}), \qquad \text{for } i \geq 1.$$

As i tends to infinity, the conditional distribution of the residual service time tends to the PH-distribution with representation $(\underline{u}, T)$, where $\underline{u}$ is the vector defined in the proof of Lemma 2.

The stationary distribution $W(\cdot)$ of the waiting time in the stable M/PH/1 queue was obtained in a particularly simple form in [2]. It is shown there that $W(\cdot)$ is the PH-distribution with representation $(\underline{\delta}, L)$, where

$$\underline{\delta} = \rho\ \underline{\pi},$$
$$L = T + \rho\ \underline{T}^{O} \cdot \underline{\pi},$$

The same method of proof as in Theorem 1 also leads to the explicit solution of the bounded queue M/PH/1/K+1, in which all customers arriving, while there are K+1 customers in the system, are lost.

Theorem 2

The stationary probability vector $\underline{x} = \left[x_0, \underline{x}_1, \ldots, \underline{x}_{K+1}\right]$ of the generator Q for the M/PH/1/K+1 queue is given by

$$\underline{x}_i = x_0\ \underline{\alpha}\ R^i, \qquad\qquad \text{for}\ \ 1 \le i \le K,$$

$$\underline{x}_{K+1} = x_0\ \underline{\alpha}\ R^K(-\lambda T^{-1}),$$

where the matrix R is given in Formula (6) and x_0 is given by

$$x_0 = \left\{ \underline{\alpha} \left[\sum_{i=0}^{K} R^i - \lambda R^K\ T^{-1} \right] \underline{e} \right\}^{-1}.$$

Remarks

a. The sum $\sum_{i=0}^{K} R^i$ in the preceding formula should not be
written in closed form, since the matrix $I - R$ may now be
singular.

b. The overflow process of an M/PH/1/K+1 queue is an example
of a Markov-modulated Poisson process. It may be informally de-
scribed as a Poisson process which is turned on only when the
Markov process Q of the M/PH/1/K+1 queue is in one of its
states $(K + 1, j)$, $1 \leq j \leq m$.
Further details and applications may be found in Neuts and Kumar
[4].

3. The PH/M/c/K+c Queue

We now consider a service system with c identical exponen-
tial servers, each of rate μ. The arrival process is now a
renewal process of phase type, whose interarrival time distribution
F(·) is of phase type with representation $(\underline{\alpha}, T)$. There are only
K waiting spaces; any customers arriving, while there are K + c
customers in the system, are lost.

This queueing model may be studied as a Markov process Q
on the state space $E = \{(i, j), 0 \leq i \leq K + c, 1 \leq j \leq m\}$. The
index i is now the number of customers in the queue and j is
the phase of the interarrival process.

The generator Q is given by

$$
Q = \begin{vmatrix}
T & \underline{T}^O \cdot \underline{\alpha} & 0 & 0 & \cdots \\
\mu I & T-\mu I & \underline{T}^O \cdot \underline{\alpha} & 0 & \cdots \\
0 & 2\mu I & T-2\mu I & \underline{T}^O \cdot \underline{\alpha} & \cdots \\
0 & 0 & 3\mu I & T-3\mu I & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
& & & & c\mu I & T-c\mu I & \underline{T}^O \underline{\alpha} \\
& & & & 0 & c\mu I & T-c\mu I & \underline{T}^O \cdot \underline{\alpha} \\
& & & & 0 & 0 & c\mu I & T+\underline{T}^O \cdot \underline{\alpha}-c\mu I
\end{vmatrix} .
$$

The stationary probability vector $\underline{x}$, partitioned into $K + c + 1$ m-vectors $\underline{x}_0$, $\underline{x}_1$, ..., $\underline{x}_{K+c}$, satisfies the equations

$$\underline{x}_0 \, T + \mu \, \underline{x}_1 = 0$$

$$(\underline{x}_{i-1} \, \underline{T}^O) \, \underline{\alpha} + \underline{x}_i \, (T - i\mu I) + (i + 1) \, \mu \, \underline{x}_{i+1} = \underline{0}, \text{ for } 1 \le i \le c - 1,$$

$$(\underline{x}_{i-1} \, \underline{T}^O) \, \underline{\alpha} + \underline{x}_i \, (T - c\mu I) + c\mu\underline{x}_{i+1} = \underline{0}, \text{ for } c \le i \le K + c - 1,$$

(11)
$$(\underline{x}_{K+c-1} \, \underline{T}^O) \, \underline{\alpha} + \underline{x}_{K+c} \, (T + \underline{T}^O \cdot \underline{\alpha} - c\mu I) = \underline{0},$$

$$\sum_{i=0}^{K+c} \underline{x}_i \, \underline{e} = 1.$$

The same device as in Theorem 1 leads to

$$(12) \quad \underline{x}_{i-1} \, \underline{T}^O = \min(i, \, c) \, \mu\underline{x}_i\underline{e}, \qquad \text{for } 1 \le i \le K + c.$$

Substitution into the equations (12), leads to

$$\underline{x}_0 = -\mu \underline{x}_1 T^{-1},$$

$$\underline{x}_i = (i+1)\mu \underline{x}_{i+1} \, (i\mu I - i\mu \underline{e} \cdot \underline{\alpha} - T)^{-1}, \qquad \text{for } 1 \le i \le c-1$$

$$\underline{x}_i = c\mu \underline{x}_{i+1} \, (c\mu I - c\mu \underline{e} \cdot \underline{\alpha} - T)^{-1}, \qquad \text{for } c \le i \le K+c-1,$$

$$\underline{x}_{K+c} = (\underline{x}_{K+c} \underline{T}^O) \, \underline{\alpha} \, (c\mu I - c\mu \underline{e} \cdot \underline{\alpha} - T)^{-1},$$

$$\sum_{i=0}^{K+c} \underline{x}_i \underline{e} = 1.$$

We now set

$$\hat{R}(i) = i\mu(i\mu I - i\mu \underline{e} \cdot \underline{\alpha} - T)^{-1}, \qquad \text{for } 1 \le i \le c$$

and

$$\gamma = (c\mu)^{-1} \, \underline{x}_{K+c} \underline{T}^O,$$

The following theorem then holds.

Theorem 3

The vector $\underline{x}$ is given by

$$\underline{x}_0 = \gamma \, c \, \underline{\alpha} \, \hat{R}(c)^{K+1} \prod_{\nu=1}^{c-1} \hat{R}(c-\nu) \, (-\mu T^{-1}),$$

$$\underline{x}_i = \gamma \, \frac{c}{i} \, \underline{\alpha} \, \hat{R}(c)^{K+1} \prod_{\nu=1}^{c-i} \hat{R}(c-\nu), \qquad \text{for } 1 \le i \le c-1,$$

$$\underline{x}_i = \gamma \, \underline{\alpha} \, \hat{R}(c)^{K+c-i+1}, \qquad \text{for } c \le i \le K+c.$$

The constant γ is uniquely determined by the normalizing equation.

Remarks

The probability that i, $1 \leq i \leq K + c$, customers are present immediately after an arrival is given by

$$\frac{\underline{x}_i \, \underline{T}^0}{\sum_{\nu=0}^{K+c} \underline{x}_\nu \underline{T}^0} = \mu_1' \, \underline{x}_i \, \underline{T}^0,$$

since by adding up the equations (11), we obtain

$$\sum_{\nu=0}^{K+c} \underline{x}_\nu \, (T + \underline{T}^0 \cdot \underline{\alpha}) = \underline{0}, \qquad \sum_{\nu=0}^{K+c} \underline{x}_\nu \, \underline{e} = 1.$$

It follows that $\displaystyle\sum_{\nu=0}^{K+c} \underline{x}_\nu = \underline{\pi}$, so that $\displaystyle\sum_{\nu=0}^{K+c} \underline{x}_\nu \, \underline{T}^0 = {\mu_1'}^{-1}$.

The stationary distributions of the waiting time at arrivals and at an arbitrary time are easily computed, once the marginal stationary densities of the queue length at arrivals and at an arbitrary time are known.

4. The PH/M/c Queue

The unbounded PH/M/c queue may be studied as a Markov process, whose generator Q is the obvious analogue of the matrix for the bounded case. This queueing model does not have a fully

explicit steady-state probability vector. It is however a simple example of a Markov process, which in the ergodic case has a modified matrix-geometric stationary vector $\underline{x}$. The vector $\underline{x}$, partitioned into m-vectors $\underline{x}_i$, $i \geq 0$, is of the form

$$\underline{x}_0, \; \underline{x}_1, \; \ldots, \; \underline{x}_{c-1}, \; \underline{x}_{c-1}R, \; \underline{x}_{c-1}R^2 \ldots \; .$$

The matrix R is now the minimal nonnegative solution to the matrix-quadratic equation

$$(13) \quad c\mu R^2 + R(T - c\mu I) + \underline{T}^o \cdot \underline{\alpha} = 0.$$

The matrix R has its maximal eigenvalue less than one, if and only if $c\mu > \mu_1'^{-1}$. This is the usual equilibrium condition for the GI/M/c queue.

The vectors $\underline{x}_0, \; \ldots, \; \underline{x}_{c-1}$, satisfy the equations

$$\underline{x}_0 T + \mu \underline{x}_1 = \underline{0},$$

$$(14) \quad (\underline{x}_{i-1} \underline{T}^o) \underline{\alpha} + \underline{x}_i (T - i\mu I) + (i + 1) \mu \underline{x}_{i+1} = \underline{0}, \text{ for } 1 \leq i \leq c-2,$$

$$(\underline{x}_{c-2} \underline{T}^o) \underline{\alpha} + \underline{x}_{c-1} \left[T - (c-1) \mu I + c\mu R \right] = \underline{0},$$

$$\sum_{i=0}^{c-2} \underline{x}_i \underline{e} + \underline{x}_{c-1} (I - R)^{-1} \underline{e} = 1.$$

By postmultiplying by $\underline{e}$ in (13) and recalling that $I - R$ is nonsingular, we obtain

$$(15) \quad c\mu R \underline{e} = \underline{T}^o.$$

By the same device as in the proof of Theorem 1, we may transform the equations (14) for $c \geq 2$, into

$$\underline{x}_0 = - \mu \, \underline{x}_1 \, T^{-1},$$

$$\underline{x}_i = (i+1) \, \mu \, \underline{x}_{i+1} \, (i\mu I - i\mu \underline{e} \cdot \underline{\alpha} - T)^{-1}, \quad \text{for } 1 \leq i \leq c-2$$

$$\underline{x}_{c-1} \left[(c-1)\mu I - (c-1) \, \mu \underline{e} \cdot \underline{\alpha} - T - c\mu R \right] = \underline{0},$$

$$\sum_{i=0}^{c-2} \underline{x}_i \underline{e} + \underline{x}_{c-1} \, (I - R)^{-1} \, \underline{e} = 1.$$

The penultimate equation determines the vector $\underline{x}_{c-1}$ up to a multiplicative constant. It is now obvious how all the vectors $\underline{x}_0, \ldots, \underline{x}_{c-1}$, may be uniquely determined.

The preceding statements all follow by applying general theorems proved in [5]. We shall limit our discussion here to that of some computational aspects.

The matrix R is computed by successive substitutions in the equation

$$R = R^2 \, c\mu (c\mu I - T)^{-1} + \underline{T}^o \cdot \underline{\alpha} \, (c\mu I - T)^{-1}.$$

It is readily seen that $T_j^o = 0$, implies that the j-th row of R is zero. The rows with index j such that $T_j^o > 0$, are strictly positive.

The latter observation is very useful, when m is large, but the vector $\underline{T}^{0}$ has few positive entries. In the case of the $E_m/M/c$ queue, for example, one may readily verify that the only non-zero row of R is the m-th row, given by

$$R_{mj} = (R_{m1})^j, \qquad \text{for } 1 \le j \le m.$$

The quantity $u_m = (R_{m1})^m$, is the unique root in (0, 1) of the equation

$$z = \left(\frac{\lambda}{\lambda + c\mu - c\mu z} \right)^m .$$

The vectors $\underline{x}_0, \ldots, \underline{x}_{c-1}$, may be computed as indicated above. In rare cases, where this recursive method of solution leads to overflow, the equations (16) may be solved iteratively, using the normalizing equation to keep the successive iterates within a compact set.

The stationary distribution $\tilde{W}(\cdot)$ of the virtual waiting time is readily seen to be given by

$$(17) \quad \tilde{W}(x) = \sum_{i=0}^{c-1} \underline{x}_i \cdots \underline{x}_{c-1} R \exp\left[c\mu x (I - R) \right] \underline{e}$$

$$\cdots \underline{x}_{c-1} R (I - R)^{-1} \cdots \underline{x}_{c-1} \exp\left[c\mu x (I - R) \right] \underline{e}^0$$

$$= 1 - \underline{x}_{c-1} (I - R)^{-1} \exp\left[-c\mu x (I - R) \right] \underline{e},$$

for $x \geq 0$. It may easily be computed by numerical solving the
differential equations

$$\underline{v}'(x) = \underline{v}(x) \, c\mu(R - I),$$

$$\underline{v}(0) = \underline{x}_{c-1} \left(I - R\right)^{-1}.$$

Similar equations may be derived for the distribution of the
waiting time at arrivals.

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Severl simple queueing models, which are commonly used as text book examples, have explicit steady-state solutions, which to date have escaped notice. These solutions are easily computable. In addition to their didactic interest, the results, presented here, are also useful in the analysis of more complex queueing models.

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